Origin of degree correlations in the Internet and other networks

Juyong Park[1,2] and M. E. J. Newman[1,3]
[1]*Department of Physics, University of Michigan, Ann Arbor, Michigan 48109, USA*
[2]*Michigan Center for Theoretical Physics, University of Michigan, Ann Arbor, Michigan 48109, USA*
[3]*Center for the Study of Complex Systems, University of Michigan, Ann Arbor, Michigan 48109, USA*
(Received 25 March 2003; revised manuscript received 6 May 2003; published 15 August 2003)

It has been argued that the observed anticorrelation between the degrees of adjacent vertices in the network representation of the Internet has its origin in the restriction that no two vertices have more than one edge connecting them. Here, we propose a formalism for modeling ensembles of graphs with single edges only and derive values for the exponents and correlation coefficients characterizing them. Our results confirm that the conjectured mechanism does indeed give rise to correlations of the kind seen in the Internet, although only a part of the measured correlation can be accounted for in this way.

DOI: 10.1103/PhysRevE.68.026112 PACS number(s): 89.75.Hc, 89.20.Hh, 05.90.+m, 64.10.+h

I. INTRODUCTION

The statistical properties of networks have been the topic of considerable attention in the physics literature in recent years [1–4]. Motivated by the availability of large-scale structural data for networks including the Internet, the World Wide Web, and social and biological networks of various kinds, researchers have created a wide selection of models of networks and processes taking place on networks. One topic of particular interest at present is the issue of degree correlations in networks. A network or graph is in general composed of some set of nodes or "vertices" joined together by lines or "edges," and the degree of a vertex is defined to be the number of edges connected to the vertex. It has been found that for many real-world networks the degrees of the vertices at either end of an edge are not independent, but are correlated with one another, either positively or negatively [5–7]. A network in which the degrees of adjacent vertices are positively correlated is said to show assortative mixing by degree, whereas a network in which they are negatively correlated is said to show disassortative mixing. A striking pattern that emerges when networks of different types are compared is that most social networks appear to be assortatively mixed, whereas most technological and biological networks appear to be disassortative [7,8].

Of particular interest to us in this paper is the Internet. At the time of writing, the Internet forms a network of about 11 000 vertices and 32 000 edges, and, as first pointed out by Pastor-Satorras *et al.* [5], the degrees of adjacent vertices have significant anticorrelation. This can be demonstrated by calculating the mean degree $\overline{k}_v^{\mathrm{nn}}$ of the neighbors of a vertex v in the network as a function of the degree k_v of that vertex. The resulting function is found to fall off with increasing k_v, roughly as a power law $k_v^{-\nu}$ with exponent $\nu \simeq 0.5$, so that the higher the degree k_v of one vertex, the lower the mean degree of its neighbors.

In a recent paper, Maslov *et al.* [6] have proposed a possible explanation for this result. Rather than supposing the anticorrelation of vertex degrees to be the result of social or engineering constraints on the construction of data networks, they suggest a topological explanation. Using computer simulations, they show for a network of the size and degree sequence of the Internet that the requirement that there is at most one edge between any pair of vertices induces degree anticorrelations very similar to those observed. And indeed there are no double edges in the Internet, a statistically unlikely occurrence were we given complete freedom about how vertices are connected.

The physical intuition behind the suggestion of Maslov *et al.* is that the restriction to single edges causes high-degree vertices to have fewer connections between them than they would if edges were assigned purely at random, and hence there must be more connections between high-degree/low-degree vertex pairs instead. A similar mechanism could be at work in other types of networks as well, such as directed networks. The World Wide Web and foodwebs are two examples of directed networks that appear to be disassortative and usually have no double edges [23].

In this paper we study the mechanism proposed by Maslov *et al.* analytically, and demonstrate that it does indeed produce disassortative mixing by degree of precisely the type observed by Pastor-Satorras *et al.* [5]. The particular model chosen by Maslov *et al.* to test their idea turns out to be difficult to treat analytically. They studied the ensemble of all graphs with a particular degree sequence and at most one edge between any vertex pair, in which each allowed graph appears with equal probability. Calculating correlations in this ensemble requires us to enumerate binary matrices with given row and column sums. No closed-form solution for such an enumeration is known at present, despite decades of study by mathematicians [9,10]. In this paper, therefore, we take a different approach, borrowing a trick from statistical mechanics. We study an expanded "grand canonical" ensemble of graphs in which the number of edges is allowed to vary under the action of a chemical potential. As network size becomes large, the number of edges becomes narrowly peaked and the predictions of the model become similar to those of the model of Maslov *et al.*, while the calculations are far easier. (A grand canonical ensemble of graphs has also been studied recently by Dorogovtsev *et al.* [11], although using a different formalism and to a different purpose.)

For networks with power-law degree distributions, we

 ©2003 The American Physical Society

will show that indeed $\overline{k_v^{\rm nn}}$ falls off as a power of k_v and derive the value of the exponent ν. We also calculate the value of the degree correlation coefficient for adjacent vertices, which measures the amount of disassortative mixing in the network. We show that the mechanism of Maslov *et al.* can account for some, but not all, of the disassortativity seen in the Internet, suggesting that there are also other mechanisms contributing to the observed degree correlations.

II. DEFINITIONS

The classic model in the study of graphs with arbitrary degree sequences is the so-called configuration model [9,12–16], in which one specifies the degree k_v of each vertex $v = 1, \ldots, n$ in a network, which also fixes the total number of edges to be $m = \frac{1}{2}\Sigma_v k_v$. Subject to the given degree sequence, the vertices are randomly wired to one another. The combinatorics of this model are however awkward and so Chung and Lu [17] recently proposed an alternative model that is in many ways more convenient. (Models similar to that of Chung and Lu have also been introduced independently by a number of other authors [11,18,19].) As we will show, by making use of an extension of their model we can make tractable the problem of counting graphs with single edges only. The model of Chung and Lu deals with undirected networks, and we consider that case first. A fairly straightforward generalization to directed networks will be dealt with briefly.

A. The network model of Chung and Lu

In the model of Chung and Lu [17] one specifies the *desired* degrees $\tilde{k}_v$ of vertices v and then places edges between vertex pairs (v,w) with probability

$$f_{vw} = \frac{\tilde{k}_v \tilde{k}_w}{2\tilde{m}}, \tag{1}$$

where $\tilde{m} = \frac{1}{2}\Sigma_v \tilde{k}_v$ is the desired number of edges in the graph. The expected degree of vertex v is then

$$\overline{k_v} = \sum_w f_{vw} = \frac{\tilde{k}_v}{2\tilde{m}} \sum_w \tilde{k}_w = \tilde{k}_v. \tag{2}$$

Thus, the expected degree of each vertex is equal to its desired degree and the expected degree distribution is asymptotically equal to the distribution of the desired degree sequence, although any individual vertex may have a degree that differs from its desired value. [Throughout this paper, we denote desired values of quantities by a tilde (e.g., $\tilde{k}$), expected values or ensemble means by a bar (e.g., $\overline{k}$), and actual values in a particular graph by undecorated characters (e.g., k).]

However, this approach is not entirely satisfactory. For some degree distributions the probability f_{vw} can exceed 1. Physically, this means that there can be more than one edge between a pair of vertices, precisely the situation that we will want to exclude in our calculations. Chung and Lu circumvent this problem by specifying an additional constraint on the distribution of desired degrees, $\tilde{k}_v \leqslant \sqrt{2\tilde{m}}$ for all v. While this condition ensures that $f_{vw} \leqslant 1$, it is strongly violated by networks, such as the Internet, that have power-law degree distributions.

Here, therefore, we adopt an alternative strategy, and adapt the model of Chung and Lu to incorporate an explicit condition that there is only one edge between every vertex pair. As we will see, this leads to some interesting physics and, in particular, to an explanation of the origin of disassortativity.

B. Ensemble of networks with single edges

We consider explicitly an ensemble of networks in which there is only a single edge between any pair of vertices. There will be an edge between the pair (v,w) with probability f_{vw} or not with probability $1-f_{vw}$. Then, the probability of occurrence of a particular graph G can be written

$$\Gamma(G) = \prod_{(v,w)} (1-f_{vw}) \prod_{\rm edges} \frac{f_{vw}}{1-f_{vw}}, \tag{3}$$

where the first product is over all unique vertex pairs (v,w) and the second is over only those pairs between which there is an edge. For convenience, we will write $P_{vw} = f_{vw}/(1-f_{vw})$, $\Gamma_0 = \Pi_{(v,w)}(1-f_{vw})$, and define δ_{vw} to be 1 if there is an edge between v and w and zero otherwise. Then

$$\Gamma(G) = \Gamma_0 \prod_{(v,w)} P_{vw}^{\delta_{vw}}. \tag{4}$$

To progress, we need to choose a form for P_{vw} or, equivalently, for f_{vw}. We will write

$$P_{vw} = P(\lambda_v, \lambda_w), \tag{5}$$

where the *fugacity* λ_v is a real number assigned to vertex v that will control the expected degree of that vertex, in a manner similar to the desired degrees in the model of Chung and Lu [17] or the fitness variables introduced in Refs. [18–20]. For the undirected network, we expect that $P_{vw} = P_{wv}$, so that $P(\lambda_v,\lambda_w) = P(\lambda_w,\lambda_v)$ is symmetric in its arguments.

We would like all graphs with a given degree sequence to appear in our ensemble with equal probability. This is the criterion applied by Maslov *et al.* [6] in their simulations, and allows us to compare our results with theirs. As we now show, this condition is sufficient to specify the form of $P(\lambda_v,\lambda_w)$.

Suppose that we have two graphs G_1 and G_2, where G_2 is obtained from G_1 by changing the positions of two edges as shown in Fig. 1, all other edges remaining untouched. (Initially there should be no edges between A and C or between B and D.) Formally, this results in the replacement of a factor $P_{AB}P_{CD}$ in $\Gamma(G_1)$, Eq. (4), by $P_{AC}P_{BD}$ to give $\Gamma(G_2)$. Since the degree sequence is invariant under this transformation, we must have $\Gamma(G_1) = \Gamma(G_2)$, and hence

$$P(\lambda_A,\lambda_B)P(\lambda_C,\lambda_D) = P(\lambda_A,\lambda_C)P(\lambda_B,\lambda_D). \tag{6}$$

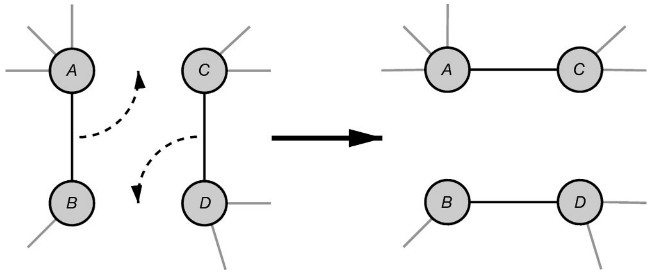

FIG. 1. The edge interchange process employed in the argument of Sec. II B. This process cannot affect the probability Γ of a graph since the degree sequence is unchanged.

Rearranging, we then find that

$$\frac{P(\lambda_A,\lambda_B)}{P(\lambda_A,\lambda_C)}=\frac{P(\lambda_D,\lambda_B)}{P(\lambda_D,\lambda_C)}, \tag{7}$$

where we have made use of the symmetry of P.

Since λ_A and λ_D each appear on only one side of this equation, it follows that both sides must be independent of both these quantities, and hence

$$\frac{P(\lambda_A,\lambda_B)}{P(\lambda_A,\lambda_C)}=\frac{P(\lambda_D,\lambda_B)}{P(\lambda_D,\lambda_C)}=\frac{g(\lambda_B)}{g(\lambda_C)}, \tag{8}$$

where $g(\lambda)$ is some function, as yet unspecified. It then follows that $P(\lambda_v,\lambda_w)$ must be factorizable in the form

$$P(\lambda_v,\lambda_w)=g(\lambda_v)g(\lambda_w). \tag{9}$$

We can confirm that the probability $\Gamma(G)$ of a graph G generated according to such a choice does indeed depend only on the degree sequence by observing that

$$\Gamma(G)=\Gamma_0\prod_v [g(\lambda_v)]^{k_v}, \tag{10}$$

where k_v is the actual degree of v in the graph G. Since Γ_0 is a constant for given $\{\lambda_v\}$, this expression is indeed a function only of the degree sequence $\{k_v\}$.

We are still free to choose the function $g(\lambda)$ in any way we wish, but all nontrivial choices are equivalent, since they just correspond to different definitions of the fugacity λ. It makes sense to make the simplest possible choice and we choose to write $g(\lambda)=\beta^{1/2}\lambda$, so that

$$P_{vw}=\beta\lambda_v\lambda_w, \tag{11}$$

where β is a free parameter that will, as we will see, control the total number of edges in the graph. [Note that λ_v here is not the same as $\tilde{k}_v$ of Chung and Lu as in Eq. (1), although in the "classical limit" of graphs with few double edges it becomes the same. See below.]

III. PREDICTIONS AND RESULTS

We now define a grand partition function

$$Z=\sum_G \Gamma(G)=\Gamma_0\sum_{\{\delta_{vw}\}}\prod_{(v,w)} P_{vw}^{\delta_{vw}}. \tag{12}$$

Interchanging the order of sum and product this gives

$$Z=\prod_{(v,w)}\sum_{\delta_{vw}} P_{vw}^{\delta_{vw}}=\prod_{(v,w)}(1+P_{vw})=\prod_{(v,w)}(1+\beta\lambda_v\lambda_w), \tag{13}$$

where we have dropped the factor of Γ_0. (As is typically the case with partition functions, leading factors of this type cancel out of all observable quantities in the theory.)

From Eq. (10) we can now see that the expected degree $\bar{k}_v$ of vertex v will be given by

$$\bar{k}_v=\frac{\lambda_v}{Z}\frac{\partial Z}{\partial\lambda_v}=-\lambda_v\frac{\partial F}{\partial\lambda_v}, \tag{14}$$

where F is the free energy

$$F=-\ln Z=-\sum_{(v,w)}\ln(1+\beta\lambda_v\lambda_w). \tag{15}$$

Combining Eqs. (14) and (15), we then get

$$\bar{k}_v=\sum_w \frac{\beta\lambda_v\lambda_w}{1+\beta\lambda_v\lambda_w}. \tag{16}$$

The expected number of edges $\bar{m}$ is the ensemble mean of the exponent of β in the partition function, which is given by

$$\bar{m}=-\beta\frac{\partial F}{\partial\beta}=\sum_{(v,w)}\frac{\beta\lambda_v\lambda_w}{1+\beta\lambda_v\lambda_w}. \tag{17}$$

The mean degree of the entire system $\bar{z}$ is simply $2\bar{m}/n$, where n is the total number of vertices.

There are clear parallels between these results and the familiar Fermi ensemble of elementary statistical mechanics. The quantity f_{vw} introduced earlier, which we can now write in the form

$$f_{vw}=\frac{\beta\lambda_v\lambda_w}{1+\beta\lambda_v\lambda_w}, \tag{18}$$

lies strictly in the range from 0 to 1, and represents the probability that an edge lies between a particular pair of vertices. This is the equivalent of the Fermi function of statistical mechanics.

The mean sum of the degrees of the neighbors of a vertex v, which we denote $\bar{K}_v^{\rm nn}$, is given by

$$\bar{K}_v^{\rm nn}=\sum_w f_{vw}\bar{k}_w=\sum_w \frac{\beta\lambda_v\lambda_w}{1+\beta\lambda_v\lambda_w}\bar{k}_w, \tag{19}$$

with $\bar{k}_w$ given by Eq. (16), and the mean degree of a neighbor of v is equal to $\bar{k}_v^{\rm nn}=\bar{K}_v^{\rm nn}/\bar{k}_v$. We will also want to calculate the correlation coefficient of the degree of vertices at either end of an edge [7], whose value is given by

$$r=\frac{\sum_v \bar{k}_v \bar{K}_v^{\mathrm{nn}}-(2\bar{m})^{-1}\left[\sum_v \bar{k}_v^2\right]^2}{\sum_v \bar{k}_v^3-(2\bar{m})^{-1}\left[\sum_v \tilde{k}_v^2\right]^2}. \tag{20}$$

Although in this paper we are dealing primarily with undirected networks, generalization of the theory to directed networks is straightforward. If f_{vw} denotes the probability of existence of a directed edge from v to w and P_{vw} is defined as before, then the expected out-degree (number of outgoing edges) of a vertex v will be

$$\bar{k}_v^{\mathrm{out}}=\sum_w f_{vw}=\sum_w \frac{P_{vw}}{1+P_{vw}}, \tag{21}$$

the expected in-degree (number of incoming edges) will be

$$\bar{k}_v^{\mathrm{in}}=\sum_w f_{wv}=\sum_w \frac{P_{wv}}{1+P_{wv}}, \tag{22}$$

and the obvious generalizations of Eqs. (17) and (19) apply.

A. Example: Power-law degree distribution

We are here interested in the case of the Internet, which, like a number of other networks, has a degree distribution that approximately follows a power law

$$p_k \propto k^{-\tau}, \tag{23}$$

with $\tau \sim 2.2 \pm 0.3$ [21,22]. The long tail of the power law means that the highest-degree vertex pairs in the network would be quite likely to have more than one edge running between them if the edges were assigned at random, and the behavior of the network changes substantially when these multiple edges are disallowed. This is the origin of the effects observed by Maslov *et al.* [6] in their simulations.

As we now show, the power-law degree distribution can be reproduced in our model by choosing the fugacity λ also to have a power-law distribution with the same exponent τ, so that the number of vertices with fugacity between λ and $\lambda+d\lambda$ is $p(\lambda)\,d\lambda$, where

$$p(\lambda)=\begin{cases} C\lambda^{-\tau} & \text{for } \lambda \geqslant \lambda_0 \\ 0 & \text{for } \lambda < \lambda_0. \end{cases} \tag{24}$$

The lower cutoff makes the distribution normalizable, and C is a normalizing constant given by

$$C^{-1}=\int_{\lambda_0}^{\infty}\lambda^{-\tau}\,d\lambda=\frac{\lambda_0^{-\tau+1}}{\tau-1}. \tag{25}$$

(One should keep in mind that λ is not restricted, as the degree is, to integer values.)

Let us consider the case $\tau=\frac{5}{2}$, for which the expressions for the quantities of interest take particularly simple forms. For this choice, the expected degree $\overline{k(\lambda)}$ of a vertex with fugacity λ is

$$\begin{aligned}\overline{k(\lambda)}&=n\int_{\lambda_0}^{\infty}\frac{\beta\lambda\lambda'}{1+\beta\lambda\lambda'}\,p(\lambda')\,d\lambda'\\ &=3n\{\beta\lambda_0\lambda-(\beta\lambda_0\lambda)^{3/2}\arctan[(\beta\lambda_0\lambda)^{-1/2}]\},\end{aligned} \tag{26}$$

and the mean degree $\bar{z}$ of the system is

$$\begin{aligned}\bar{z}&=\frac{2\bar{m}}{n}=2\int_{\lambda_0}^{\infty}\overline{k(\lambda)}\,p(\lambda)d\lambda\\ &=9n\beta\lambda_0^2\left[1-\frac{1}{4}\Phi\left(-\frac{1}{\beta\lambda_0^2},2,\frac{1}{2}\right)\right],\end{aligned} \tag{27}$$

where $\Phi(x,a,b)$ denotes the analytic continuation of the Lerch transcendent.

The parameter β is to some extent redundant in these expressions, since we are free to choose λ as we wish, but it proves convenient nonetheless. If we choose $\beta=(2\tilde{m})^{-1}$, where $\tilde{m}$ is the desired number of edges as before, then for graphs in which there are few double edges we have $f_{vw} \simeq (\lambda_v\lambda_w)/(2\tilde{m})$, giving $\bar{k}_v=\Sigma_w f_{vw} \simeq \Sigma_w(\lambda_v\lambda_w)/(2\tilde{m}) \simeq \lambda_v$, so that the fugacity is simply equal to the desired degree of a vertex, as in the model of Chung and Lu [17].

The regime in which there are few double edges can be thought of as the classical limit of our Fermi ensemble, and corresponds to the case where the first terms in Eqs. (26) and (27) dominate. As λ becomes large, however, encouraging vertices to have a high degree, we enter the quantum regime, where it becomes harder and harder for vertices to find others to connect to. This is reflected in Eq. (26) also. Expanding the inverse tangent as $\arctan x = x-\frac{1}{3}x^3+\frac{1}{5}x^5-O(x^7)$, we find that the leading term cancels and

$$\overline{k(\lambda)}=n-\frac{3n}{5\beta\lambda_0\lambda}+nO[(\beta\lambda_0\lambda)^{-2}]. \tag{28}$$

Thus, as $\lambda\to\infty$ the degree tends to n, as we would expect, since this is the largest degree a vertex can have on a network with no double edges.

The mean sum $\overline{K^{\mathrm{nn}}}(\lambda)$ of the degrees of the neighbors of a vertex with fugacity λ is

$$\begin{aligned}\overline{K}^{\rm nn}(\lambda) &= n\int_{\lambda_0}^{\infty}\frac{\beta\lambda\lambda'}{1+\beta\lambda\lambda'}\overline{k(\lambda')}\,p(\lambda')\,d\lambda' \\ &= 9n^2(\beta\lambda_0\lambda)^{3/2}\Bigg[\frac{\lambda_0}{\lambda}\arctan[(\beta\lambda_0\lambda)^{-1/2}] \\ &\quad -\frac{\pi}{4}\left(\frac{\lambda_0}{\lambda}\right)^{3/2}\{2\ln[1+(\lambda/\lambda_0)^{1/2}]-\ln(1+\beta\lambda_0\lambda) \\ &\quad +O(\lambda_0\beta^{1/2})\}\Bigg], \end{aligned} \tag{29}$$

and from this we can calculate $\overline{k}^{\rm nn}$ [24].

These results can be extended to other values of τ also, although the formulas are not as elegant as for the case $\tau=\frac{5}{2}$. For example, for general $\tau>1$ the equivalent of Eq. (26) is

$$\overline{k(\lambda)} = n\,{}_2F_1\left(1,-1+\tau;\tau;-\frac{1}{\beta\lambda_0\lambda}\right), \tag{30}$$

where ${}_2F_1$ is a hypergeometric function. This form is used in some of the calculations in the following section.

B. Comparison with the Internet

We now compare our model quantitatively with the Internet graph. To do this, it is important that we make the size n and the number of edges $\tilde{m}$ the same as the real Internet, since our predictions, Eqs. (26) and (29), are dependent on these quantities. For the purposes of comparison, we use the data of Chen *et al.* [22] from 2001 on the structure of the Internet at the autonomous system level, for which $n=10\,697$ and $\tilde{m}=31\,992$, which gives a mean degree of $\overline{z}=2\tilde{m}/n=5.981$. For the choice Eq. (24) of fugacity distribution used here, we can arrange for the network to have the correct mean degree by an appropriate choice of the lower limit λ_0 of the distribution, and we do this for three values $\tau=2.1$, 2.3, and 2.5 of the exponent of the power law. We also perform extensive simulations of the model for the same parameter values to confirm our calculations, and analytic and numerical results are shown in Figs. 2–4 and in Table I. As we can see, analytical and numerical predictions agree closely.

Consider first Fig. 2, which is a plot of the mean degree of a vertex as a function of its fugacity. As the figure shows, the degree is closely linear in the fugacity for small λ and flattens off as degree approaches n, as expected.

The same behavior is evident in Fig. 3 also, which shows the cumulative distribution function of degrees in simulations of the model for power-law distributed fugacity, Eq. (24). The distribution of degrees also follows a power law (a straight line on the logarithmic axes used), until the degree approaches n, where the distribution is cut off. This is an eminently sensible behavior: given the constraint of single edges only, presumably the real Internet must deviate from the power-law behavior for large degree, and our model should and does reflect this behavior.

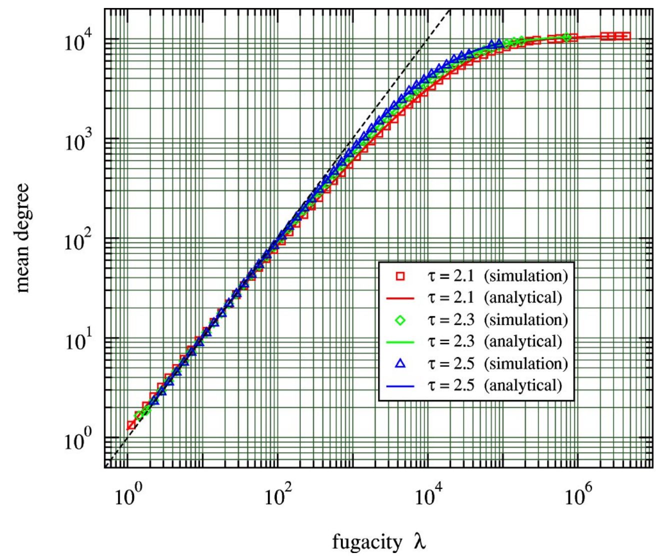

FIG. 2. The ensemble mean $\overline{k}$ of the degree of a vertex in our model as a function of the fugacity λ of the vertex. The numerical results are averaged over 1000 repetitions of the simulation. The dotted line indicates the form $\overline{k}=\lambda$, which the curve is expected to approximate for small λ.

The fundamental result of this paper is shown in Fig. 4, where we have plotted the mean degree $\overline{k}^{\rm nn}$ of the neighbors of a vertex, calculated from Eq. (29), against the degree of that vertex. This is the comparison used by Pastor-Satorras *et al.* [5] to demonstrate degree anticorrelation in the Internet. As the figure shows, there is a clear decline in the value of $\overline{k}^{\rm nn}$ as degree increases, just as in the real Internet, con-

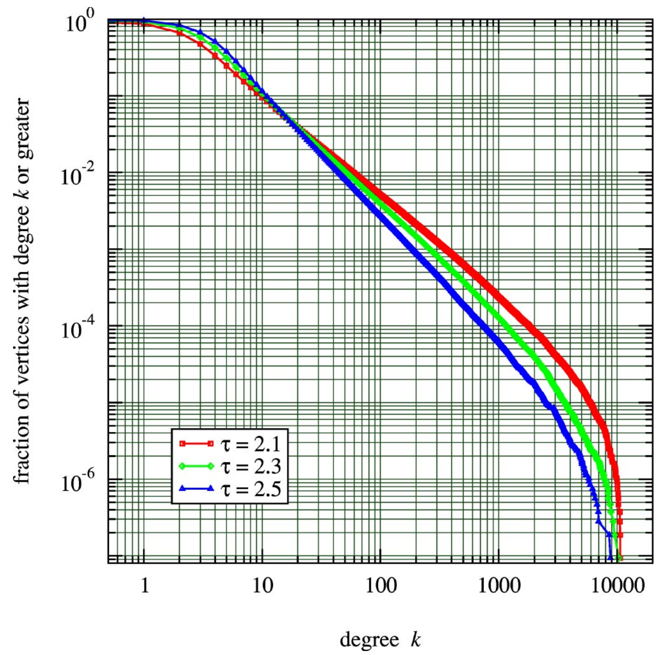

FIG. 3. The cumulative distribution function for vertex degree in simulations of our model. The general form of the distribution is a power law for low degree with a cutoff as degree approaches the system size n.

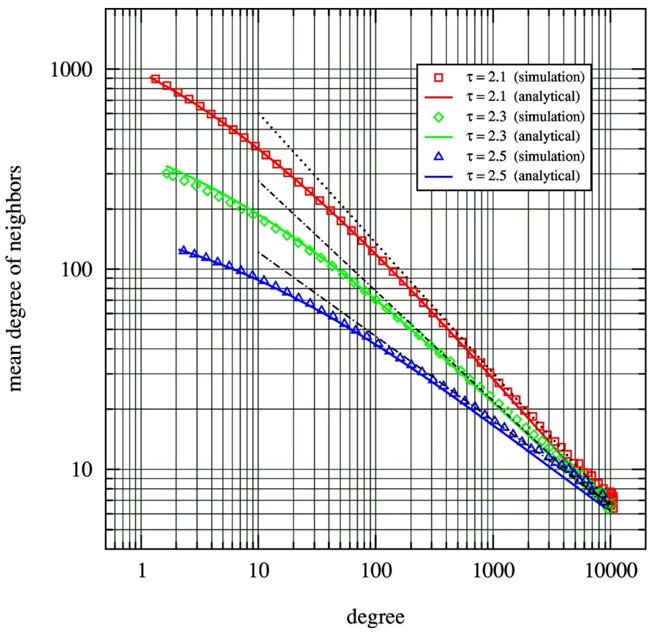

FIG. 4. The mean degree $\overline{k}^{\mathrm{nn}}$ of the neighbors of a vertex as a function of the degree $\overline{k}$ of that vertex. The dotted lines show the asymptotic slopes of the curves.

firming that the single-edge constraint does indeed give rise to anticorrelations, as conjectured by Maslov *et al.* Furthermore, the decline appears to be approximately power-law in form $\overline{k}^{\mathrm{nn}} \sim \overline{k}^{-\nu}$, as found by Pastor-Satorras *et al.* We can deduce approximate values for the exponent ν from our results. We find for $\tau = 2.1$, $\nu \simeq 0.65$, for $\tau = 2.3$, $\nu \simeq 0.55$, and for $\tau = 2.5$, $\nu \simeq 0.42$. The slopes are shown as the dotted lines in Fig. (3). The values for ν are all close to the value $\nu \simeq 0.5$ observed for the real Internet [5]. The power law is only approximate however—the functional form of Eq. (29) is not just a simple power law, and we can see from the figure that the slope of $\overline{k}^{\mathrm{nn}}$ is smaller for smaller $\overline{k}$. The same behavior is visible in both the real Internet data and the simulation results of Maslov *et al.* [6].

Finally, in Table I, we show values for the mean degree $\overline{z}$ and degree correlation coefficient r for our model. As we see, the theoretical calculations and numerical results again agree well. Since the Internet is disassortative, we expect the degree correlation coefficient to be negative in the real network, and its value has been measured to be $r = -0.189$ [7]. In the model we also see negative values of r, whose magnitude depends quite strongly on the value of the exponent τ. A detailed comparison of model and real-world data may therefore have to wait on more precise measurements of the degree distribution (about which there is at present some dispute [22]). However, it is interesting to note that none of the cases in Table I is as strongly anticorrelated as the real Internet. Thus, our calculations appear to indicate that some of the disassortativity in the Internet can be accounted for by the mechanism proposed by Maslov *et al.*, but probably not all of it. The remainder of the disassortativity is presumably due to engineering or social constraints on the structure of the network. One possibility, which has been discussed elsewhere [6,8], is that the Internet is divided into connectivity providers such as phone companies and Internet service providers, who tend to have high degree, since they have lots of customers, and end users of connectivity, who typically have a degree of only one or two. Most connections in the network run from the providers to the end users and are therefore from high to low degree, giving a social reason for disassortativity in addition to the purely topological one considered here.

TABLE I. Mean degree and degree correlation coefficient for the networks generated by our model from both the analytic theory and from computer simulations. The simulation results are averaged over 1000 networks each. Figures in parentheses show statistical errors on the least significant figures.

	Mean degree $\overline{z}$		Degree correlation r	
τ	Theory	Simulation	Theory	Simulation
2.1	5.981	5.982(15)	-0.0950	$-0.0932(17)$
2.3	5.981	5.972(9)	-0.0541	$-0.0551(18)$
2.5	5.981	5.986(7)	-0.0304	$-0.0321(14)$

IV. DISCUSSION AND CONCLUSIONS

In this paper we have studied analytically ensembles of networks where there is at most one edge between any pair of vertices. By making use of an enlarged ensemble in which the number of edges is allowed to vary in a manner reminiscent of the Fermi ensemble of traditional statistical mechanics, we have been able to find closed-form expressions for ensemble averages of a number of quantities of interest. In particular, we have confirmed the previous numerical finding [6] that graph ensembles with single edges have negative correlations between the degrees of adjacent vertices. This has been proposed as a possible explanation for the anticorrelation or disassortativity observed in the topology of the Internet [5]. We find that the restriction to single edges can account for some but not all of the correlations observed in real Internet data.

The same mechanism could be responsible for disassortativity in other networks also. Many networks, including citation networks, the World Wide Web, social networks, collaboration networks, metabolic and genetic regulatory networks, and food webs have, at least in their most common representations, only single edges between vertex pairs. Thus, it is reasonable to suppose that these networks would be disassortative also, and indeed this appears to be the case for most networks that have been studied [8]. There is one important exception to this rule however: almost all social networks, appear to be significantly *assortative* in their mixing patterns. We conjecture, therefore, that disassortativity by degree is the normal state of affairs for a network, as a result of the mechanisms described in this paper, with social networks being assortative probably because of additional social effects that are absent from other network types; for one reason or another, it appears that gregarious people prefer to associate with other gregarious people. Furthermore, when assessing the level of assortativity in a social network, one should take into account the natural tendency for networks to be disassortative, since this tendency implies that to reach a level even of neutral assortativity would take a moderately strong bias in favor of positive degree correlation, and reaching a substantially assortative state would take a very strong such bias.

Finally, we point out that the general analytical technique employed in this paper, of enlarging an ensemble, of graphs to create a grand canonical ensemble, may have applications to other problems in the study of networks also. It is well known among statistical physicists that using such an ensemble often makes the analytic treatment of a problem easier, and the results presented here offer hope that this approach may prove useful in other settings.

ACKNOWLEDGMENTS

The authors thank Supriya Krishnamurthy for useful discussions and comments. This work was supported in part by the National Science Foundation under Grant No. DMS-0234188 and by the Michigan Center for Theoretical Physics at the University of Michigan.

[1] S.H. Strogatz, Nature (London) **410**, 268 (2001).
[2] R. Albert and A.-L. Barabási, Rev. Mod. Phys. **74**, 47 (2002).
[3] S.N. Dorogovtsev and J.F.F. Mendes, Adv. Phys. **51**, 1079 (2002).
[4] M.E.J. Newman, SIAM Rev. **45**, 167 (2003).
[5] R. Pastor-Satorras, A. Vázquez, and A. Vespignani, Phys. Rev. Lett. **87**, 258701 (2001).
[6] S. Maslov, K. Sneppen, and A. Zaliznyak, e-print cond-mat/0205379.
[7] M.E.J. Newman, Phys. Rev. Lett. **89**, 208701 (2002).
[8] M.E.J. Newman, Phys. Rev. E **67**, 026126 (2003).
[9] E.A. Bender and E.R. Canfield, J. Comb. Theory, Ser. A **24**, 296 (1978).
[10] B.-Y. Wang and F. Zhang, Discrete Math. **187**, 211 (1998).
[11] S.N. Dorogovtsev, J.F.F. Mendes, and A.N. Samukhin, e-print cond-mat/0204111.
[12] B. Bollobás, Eur. J. Comb. **1**, 311 (1980).
[13] T. Łuczak, in *Proceedings of the Symposium on Random Graphs, Poznań 1989*, edited by A.M. Frieze and T. Łuczak (Wiley, New York, 1992), pp. 165–182.
[14] M. Molloy and B. Reed, Random Struct. Algorithms **6**, 161 (1995).
[15] M. Molloy and B. Reed, Combinatorics, Probab. Comput. **7**, 295 (1998).
[16] M.E.J. Newman, S.H. Strogatz, and D.J. Watts, Phys. Rev. E **64**, 026118 (2001).
[17] F. Chung and L. Lu, Annals Combinatorics **6**, 125 (2002).
[18] K.-I. Goh, B. Kahng, and D. Kim, Phys. Rev. Lett. **87**, 278701 (2001).
[19] G. Caldarelli, A. Capocci, P. De Los Rios, and M.A. Muñoz, Phys. Rev. Lett. **89**, 258702 (2002).
[20] G. Bianconi and A.-L. Barabási, Phys. Rev. Lett. **86**, 5632 (2001).
[21] M. Faloutsos, P. Faloutsos, and C. Faloutsos, Comput. Commun. Rev. **29**, 251 (1999).
[22] Q. Chen, H. Chang, R. Govindan, S. Jamin, S.J. Shenker, and W. Willinger, in *Proceedings of the 21st Annual Joint Conference of the IEEE Computer and Communications Societies* (IEEE Computer Society, New York, 2002).
[23] The Web can have more than one link from one page to another, but in most studies of the topology of the Web graph duplicate links have been neglected.
[24] To be precise, we note that our formulas give $\overline{k}^{\mathrm{nn}}$ as a function of the expected degree of a vertex, rather than its actual degree. However in the large-degree tail, where the power-law behavior of interest is observed, we expect that actual degree will be narrowly peaked around the expected value.